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Donsker's Delta Function of the Generalized Mixed Fractional Brownian Motion

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Abstract

The generalized mixed fractional Brownian motion is defined by taking linear combinations of a finite numbers of independent fractional Brownian motions with different Hurst parameters. It is a Gaussian process with stationary increments, posseses self-similarity property, and in general, is neither a Markov process nor a martingale. In this paper we review the generalized mixed fractional Brownian motion from the white noise analysis point of view. In several applications, for example in the context of polymer physics, Feynmans path integral and financial mathematics, we need to pin a stochastic process at a particular spatial coordinate. Motivated by this fact we study the so-called Donskers delta function of the generalized mixed fractional Brownian motion. It is defined as the informal composition of the Dirac delta function with the generalized mixed fractional Brownian motion. As a main result we prove that Donskers delta function of the generalized mixed fractional Brownian motion is a well-defined mathematical object in the space of Hida distributions. Furthermore, an explicit expression for the generalized expectation of the Hida distribution is also obtained.

1 Introduction

Cheridito in [2] introduced the concept of mixed fractional Brownian motion as a generalization of fractional Brownian motion. Let a and b be two real numbers such that $(a, b) \neq (0, 0)$. A *mixed fractional Brownian motion* (MFBM) of parameter H, a, and b is a stochastic process $M^H := (M_t^H)_{t\geq 0} := (M_t^{H,a,b})_{t\geq 0}$ defined on some probability space by $M_t^H = M_t^{H,a,b} := aB_t + bB_t^H$, where $(B_t)_{t\geq 0}$ is a Brownian motion and $(B_t^H)_{t\geq 0}$ is an independent fractional Brownian motion of Hurst parameter $H \in (0, 1)$. This process was introduced to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. Some (probabilistic and analytic) properties of MFBM was investigated by Zili in [12].

MFBM has been further generalized by Thäle in [10] to the arbitrary finite linear combinations of independent fractional Brownian motions. In this paper we shall call this process as generalized mixed fractional Brownian motion. Let $\alpha_1, \ldots, \alpha_n$, $n \in \mathbb{N}$ be real numbers and not all α_k equals zero. A *generalized mixed fractional Brownian motion* (GMFBM) of parameter $H = (H_1, \ldots, H_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a stochastic process $Z^H := (Z_t^H)_{t\geq 0} := (Z_t^{H,\alpha})_{t\geq 0}$ defined on some probability space by $Z_t^H = Z_t^{H,\alpha} := \sum_{k=1}^n \alpha_k B_t^{H_k}$, where $(B_t^{H_k})_{t\geq 0}$, $k = 1, \ldots, n$ are independent fractional Brownian motions of Hurst parameter $H_k \in (0, 1)$. Recent research in the internet traffic modeling using self-similar process shows the need of study of GMFBM, see Drakakis and Radulovic[4]. Below we collect some basic properties of the GMFBM. For proofs and further comments on the importance of this process see Thäle[10] and references therein.

Proposition 1 *[10] The GMFBM $Z^H = (Z_t^{H,\alpha})_{t\geq 0}$ is a centered Gaussian process with variance $\sum_{k=1}^n \alpha_k^2 t^{2H_k}$ and covariance function*

$$\mathbb{E}\left(Z_t^{H,\alpha} Z_s^{H,\alpha}\right) = \frac{1}{2}\sum_{k=1}^{n} \alpha_k^2 \left(t^{2H_k} + s^{2H_k} - |t-s|^{2H_k}\right).$$

Z^H has stationary increments and they are correlated if and only if $H_k = \frac{1}{2}$ for all k. Z^H is also $(c_1, \ldots, c_n; H_1, \ldots, H_n)$-self-similar in the sense that $\sum_{k=1}^n \alpha_k c_k^{-H_k} B_{c_k t}^{H_k} = \sum_{k=1}^n \alpha_k B_t^{H_k}$ in law. Z^H is neither a Markov process nor a semimartingale, unless $H_k = \frac{1}{2}$ for all k. Z^H exhibits a long-range dependence if and only if there exists k with $H_k > \frac{1}{2}$. For all $T > 0$, with probability one Z^H has a version, the sample path of which are Hölder continuous of order $\gamma < \min_{1\leq k\leq n} H_k$ on the interval $[0,T]$. Every sample path of Z^H is almost surely nowhere differentiable.

In this paper we focus on the so-called Donsker's delta function of the GMFBM. This object is used widely in many areas such as quantum mechanics, polymer physics and financial mathematics. The organization of the paper is as follow. In Section 2 we review the construction and analysis of the GMFBM in the framework of white noise analysis. Our main result concerning the Donsker delta function is the content of Section 3. As a summary we are able to give meaning to the Donsker delta function of GMFBM as a white noise distribution and moreover, we can compute its generalized expectation.

2 White Noise Analysis of GMFBM

We briefly recall some fundamental concepts of white noise analysis used throughout this paper. White noise analysis is an infinite dimensional stochastic calculus using white noise, i.e. the generalized time derivative of the Brownian motion, as the basic random variable. For a more comprehensive explanation including various applications of white noise analysis, see Hida et al[8], Kuo[7], Obata[11].

In the first part of this section we will give a representation of GMFBM as a random variable on the white noise space. Firstly, we summarize the construction of a fractional Brownian motion in the white noise space. This approach was first introduced by Bender in [1]. Details and proofs can be found in [1] and references therein. Since GMFBM is a linear combination of independent fractional Brownian motions, its realization in the white noise space can be easily derived. For this purpose we follow closely the approach developed by Drumond et al in [5]. The materials presented in this section is a slight generalization of the one in Suryawan [9].

Let $(\mathcal{S}'_d(\mathbb{R}), \mathcal{C}, \mu)$ be the vector-valued white noise space, i.e., $\mathcal{S}'_d(\mathbb{R})$ is the space of vector-valued tempered distribution, $\mathcal{C}$ is the Borel σ-algebra generated by cylinder sets in $\mathcal{S}'_d(\mathbb{R})$, and the probability measure μ is uniquely determined by the Bochner-Minlos theorem such that its characteristic function given by

$$C(\vec{f}) := \int_{\mathcal{S}'_d(\mathbb{R})} e^{i\langle\vec{\omega},\vec{f}\rangle}\, d\mu(\vec{\omega}) = e^{-\frac{1}{2}|\vec{f}|_0^2}$$

for all vector-valued Schwartz test function $\vec{f} \in \mathcal{S}_d(\mathbb{R})$. Here $|\cdot|_0$ denotes the usual norm in the real Hilbert space of all vector-valued Lebesgue square-integrable functions $L_d^2(\mathbb{R})$, and $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $\mathcal{S}'_d(\mathbb{R})$ and $\mathcal{S}_d(\mathbb{R})$. This dual pairing can be considered as the bilinear extension of the inner product on $L_d^2(\mathbb{R})$, i.e.

for all $\vec{g} = (g_1, \dots, g_d) \in L^2_d(\mathbb{R})$ and $\vec{f} = (f_1, \dots, f_d) \in \mathcal{S}_d(\mathbb{R})$. We should remark that we have the following real Gel'fand triple

$$\mathcal{S}_d(\mathbb{R}) \subset L^2_d(\mathbb{R}) \subset \mathcal{S}'_d(\mathbb{R}).$$

It is well known that in the white noise space a version of the d-dimensional Brownian motion is given by the stochastic process $(B_t)_{t\geq 0}$ with

$$B_t := \left(\left\langle \cdot, \mathbf{1}_{[0,t]}\right\rangle, \cdots, \left\langle \cdot, \mathbf{1}_{[0,t]}\right\rangle\right),$$

such that

$$B_t(\vec{\omega}) := \left(\left\langle \omega_1, \mathbf{1}_{[0,t]}\right\rangle, \cdots, \left\langle \omega_d, \mathbf{1}_{[0,t]}\right\rangle\right), \quad \vec{\omega} = (\omega_1, \dots, \omega_d) \in \mathcal{S}'_d(\mathbb{R})$$

where $\mathbf{1}_A$ denotes the indicator function on a set A.

To obtain a similar representation for fractional Brownian motion in term of indicator function one makes use the concept of Weyl's fractional integral and Marchaud's fractional derivative. Precisely speaking, for an arbitrary Hurst parameter $H \in (0,1)$, a version of a d-dimensional fractional Brownian motion is given by

$$B^H_t(\vec{\omega}) := \left(\left\langle \omega_1, N^H_- \mathbf{1}_{[0,t]}\right\rangle, \cdots, \left\langle \omega_d, N^H_- \mathbf{1}_{[0,t]}\right\rangle\right), \quad \vec{\omega} = (\omega_1, \dots, \omega_d) \in \mathcal{S}'_d(\mathbb{R})$$

where for a real-valued function f the operator N^H_- is defined by

$$N^H_- f := \begin{cases} \frac{(\frac{1}{2}-H)K_H}{\Gamma(H+\frac{1}{2})} \lim_{\varepsilon \searrow 0} \int_\varepsilon^\infty \frac{f(x)-f(x+y)}{y^{\frac{3}{2}-H}}\, dy & , \text{ if } H \in (0, \frac{1}{2}) \\ f & , \text{ if } H = \frac{1}{2} \\ \frac{K_H}{\Gamma(H-\frac{1}{2})} \int_x^\infty f(y)(y-x)^{H-\frac{3}{2}}\, dy & , \text{ if } H \in (\frac{1}{2}, 1) \end{cases}$$

provided the integrals exist for almost all $x \in \mathbb{R}$. Here Γ denotes Gamma function and K_H is the normalizing constant. Aparts from the operator N^H_-, we shall also consider the operator N^H_+ defined by

$$N^H_+ f := \begin{cases} \frac{(\frac{1}{2}-H)K_H}{\Gamma(H+\frac{1}{2})} \lim_{\varepsilon \searrow 0} \int_\varepsilon^\infty \frac{f(x)-f(x-y)}{y^{\frac{3}{2}-H}}\, dy & , \text{ if } H \in (0, \frac{1}{2}) \\ f & , \text{ if } H = \frac{1}{2} \\ \frac{K_H}{\Gamma(H-\frac{1}{2})} \int_{-\infty}^x f(y)(x-y)^{H-\frac{3}{2}}\, dy & , \text{ if } H \in (\frac{1}{2}, 1). \end{cases}$$

Thus, the following representation of GMFBM is well defined. For $H = (H_1, \dots, H_n)$, and $\alpha = (\alpha_1, \dots, \alpha_n)$, $H_k \in (0,1)$, $\alpha_k \in \mathbb{R}$, $n \in \mathbb{N}$ a version of a d-dimensional GMFBM of parameter H and α in the white noise analysis setting is given by

$$Z^{H,\alpha}_t(\vec{\omega}) := \left(\left\langle \omega_1, \sum_{k=1}^n \alpha_k N^{H_k}_- \mathbf{1}_{[0,t]}\right\rangle, \cdots, \left\langle \omega_d, \sum_{k=1}^n \alpha_k N^{H_k}_- \mathbf{1}_{[0,t]}\right\rangle\right),$$

for $\vec{\omega} = (\omega_1, \dots, \omega_d) \in \mathcal{S}'_d(\mathbb{R})$. Within this formalism we can always choose a continuous version of the process according to Kolmogorov-Centsov theorem. At this point we should also emphasize that the white noise analysis approach in defining GMFBM has the advantage that the underlying probability space does not depend on the Hurst parameter under consideration.

It is known that for some type of functions, for example $f = \mathbf{1}_{[0,t]}, t > 0$ or $f \in \mathcal{S}_1(\mathbb{R})$, N^H_- and N^H_+ are dual operators in the sense that the following integration by parts formula holds

$$\int_{\mathbb{R}} f(x) N^H_- g(x)\, dx = \int_{\mathbb{R}} (N^H_+ f)(x) g(x)\, dx.$$

Moreover, the following estimation was proved in Drumond et al[5] and will be very useful for our purpose.

Lemma 2 *[5] If $H \in (0,1)$ and $f \in \mathcal{S}_1(\mathbb{R})$, then there exists a nonnegative constant C_H, independent of f, such that*

$$\left| \int_{\mathbb{R}} f(x) N_-^H \mathbf{1}_{[s,t]}(x)\, dx \right| \leq C_H (t-s) \left(\sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\in\mathbb{R}} |f'(x)| + |f|_0 \right)$$

for all $s < t$.

In the second part of this section we will review the concept of Hida distribution and characterization theorem in white noise analysis. Consider $L^2(\mathcal{S}'_d(\mathbb{R}), \mathcal{C}, \mu)$ as a complex Hilbert space. This space is unitary isomorphic to the Fock space of symmetric square-integrable function, i.e.

$$L^2(\mu) := L^2(\mathcal{S}'_d(\mathbb{R}), \mathcal{C}, \mu) \cong \left(\bigoplus_{k=0}^{\infty} \mathrm{sym} L^2(\mathbb{R}^k, k! d^k x) \right)^{\otimes d}.$$

This will lead to the Wiener-Itô chaos expansion of an element $F \in L^2(\mu)$,

$$F(\omega_1, \ldots, \omega_d) = \sum_{(m_1,\ldots,m_d)\in\mathbb{N}_0^d} \left\langle : \omega_1^{\otimes m_1} : \otimes \ldots \otimes : \omega_d^{\otimes m_d} :, \vec{f}_{(m_1,\ldots,m_d)} \right\rangle, \tag{1}$$

with kernel functions $\vec{f}_{(m_1,\ldots,m_d)}$ of the m-th chaos in the Fock space. Here $: \omega^{\otimes n} :$ denotes the m-th Wick power of $\omega \in \mathcal{S}'_1(\mathbb{R})$, see Hida et al[8] for details. Using the Wiener-Ito chaos expansion and the second quantization operator we can construct a Gel'fand triple

$$(\mathcal{S}) \subset L^2(\mu) \subset (\mathcal{S})^*$$

where $(\mathcal{S})$ is the space of white noise test functions such that its topological dual space is the space $(\mathcal{S})^*$, the space of generalized white noise functionals. Elements of $(\mathcal{S})$ and $(\mathcal{S})^*$ are also known as Hida test functions and Hida distributions, respectively. The reader not familiar with this subject may consult Kuo [7] or Hida et al [8]. As we know a GMFBM Z^H is nowhere differentiable on almost every path. However, it is possible to show that Z^H is differentiable as a mapping from $\mathbb{R}$ into $(\mathcal{S})^*$, and the distributional derivative of $Z^H = Z_t^{H,\alpha}$ in $(\mathcal{S})^*$ is given by

$$W_t^{H,\alpha}(\vec{\omega}) := \left(\left\langle \omega_1, \sum_{k=1}^{n} \alpha_k \delta_t \circ N_+^{H_k} \right\rangle, \cdots, \left\langle \omega_d, \sum_{k=1}^{n} \alpha_k \delta_t \circ N_+^{H_k} \right\rangle \right),$$

where $\vec{\omega} = (\omega_1, \ldots, \omega_d) \in \mathcal{S}'_d(\mathbb{R})$ and δ_t denotes the Dirac delta functional at t. The Hida distribution $W_t^{H,\alpha}$ is called *generalized mixed fractional white noise*. For a detail exposition see Suryawan [9].

The rest of this section is devoted to the characterization of a Hida distribution via the so-called S-transform. For a given $\vec{f} \in \mathcal{S}_d(\mathbb{R})$ and the corresponding Wick exponential

$$: e^{\langle \vec{\omega}, \vec{f} \rangle} := C(\vec{f}) e^{\langle \vec{\omega}, \vec{f} \rangle},$$

we define the *S-transform* of an element $\Phi \in (\mathcal{S})^*$ by

$$(S\Phi)(\vec{f}) := \left\langle \left\langle \Phi, : e^{\langle \cdot, \vec{f} \rangle} : \right\rangle \right\rangle, \quad \text{for all } \vec{f} \in \mathcal{S}_d(\mathbb{R}). \tag{2}$$

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Here $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the dual pairing between $(\mathcal{S})$ and $(\mathcal{S})^*$ which is defined as the bilinear extension of the sesquilinear inner product on $L^2(\mu)$. The S-transform provides a quite useful way to identify a Hida distribution $\Phi \in (\mathcal{S})^*$, in particular, when it is very hard or impossible to find the explicit form for the Wiener-Ito chaos expansion of Φ.

Theorem 3 [6] *A function $F : \mathcal{S}_d(\mathbb{R}) \to \mathbb{C}$ is the S-transform of a Hida distribution in $(\mathcal{S})^*$ if and only if it satisfies the conditions:*

(1) F is ray analytic, i.e., for every $\vec{f}, \vec{g} \in \mathcal{S}_d(\mathbb{R})$ the mapping $\mathbb{R} \ni \lambda \mapsto F\left(\lambda\vec{f} + \vec{g}\right)$ has an entire extension to $\lambda \in \mathbb{C}$, and

(2) F has growth of second order, i.e., there exist constants $K_1, K_2 > 0$ and a continuous norm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$ such that $\left|F(z\vec{f})\right| \leq K_1 e^{K_2|z|^2\|\vec{f}\|^2}$, for all $z \in \mathbb{C}$, $\vec{f} \in \mathcal{S}_d(\mathbb{R})$.

The last theorem has an important consequence which deals with the Bochner integration of a family of Hida distributions which depend on an additional parameter. For details and proofs see [6].

Corollary 4 *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ be a mapping from Ω to $(\mathcal{S})^*$. If the S-transform of Φ_λ fulfils the following two conditions:*

(1) the mapping $\lambda \mapsto (S\Phi_\lambda)(\vec{f})$ is measurable for all $\vec{f} \in \mathcal{S}_d(\mathbb{R})$, and

(2) there exist $C_1(\lambda) \in L^1(\Omega, \nu)$, $C_2(\lambda) \in L^\infty(\Omega, \nu)$ and a continuous norm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$ such that $\left|(S\Phi_\lambda)(z\vec{f})\right| \leq c_1(\lambda) e^{C_2(\lambda)|z|^2\|\vec{f}\|^2}$, for all $z \in \mathbb{C}$, $\vec{f} \in \mathcal{S}_d(\mathbb{R})$,

then Φ_λ is Bochner integrable with respect to some Hilbertian norm which topologizing $(\mathcal{S})^*$. Hence $\int_\Omega \Phi_\lambda \, d\nu(\lambda) \in (\mathcal{S})^*$, and furthermore

$$S\left(\int_\Omega \Phi_\lambda \, d\nu(\lambda)\right)(\vec{f}) = \int_\Omega (S\Phi_\lambda)(\vec{f}) \, d\nu(\lambda).$$

3 Donsker's Delta Function

In order to "pin" GMFBM at a point $c \in \mathbb{R}^d$ we consider the Donsker's delta function of GMFBM which is defined as the informal composition of the Dirac delta distribution $\delta_d \in \mathcal{S}'(\mathbb{R}^d)$ with a d-dimensional GMFBM $\left(Z_t^H\right)_{t\geq 0}$, i.e., $\delta_d\left(Z_t^H - c\right)$. We can give a rigorous meaning to the Donskers's delta function as a Hida distribution. For this end we recall the Fourier-transform representation of Dirac delta distribution which is given by

$$\delta_d(x - c) = \frac{1}{2\pi} \int_{\mathbb{R}^d} e^{i\lambda(x-c)} \, d\lambda.$$

Now we are ready to prove the main result of this paper.

Proposition 5 *Let $c \in \mathbb{R}^d$. The Bochner integral*

$$\delta_d\left(Z_t^H - c\right) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\lambda\left(Z_t^H - c\right)} \, d\lambda,$$

is a Hida distribution with S-transform given by

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$$\left(S\delta_d\left(Z_t^H - c\right)\right)(\vec{f}) \tag{3}$$
$$= \frac{1}{\left(2\pi \sum_{k=1}^n \alpha_k^2 t^{2H_k}\right)^{d/2}} \exp\left(-\frac{1}{2\sum_{k=1}^n \alpha_k^2 |t^{2H_k}} \sum_{j=1}^d \left(\int_{\mathbb{R}} \sum_{k=1}^n \alpha_k f_j(x) N_-^{H_k} \mathbf{1}_{[0,t]}(x)\, dx\right)^2\right),$$

for all $\vec{f} = (f_1, \ldots, f_d) \in \mathcal{S}_d(\mathbb{R})$.

PROOF. For $\vec{f} \in \mathcal{S}_d(\mathbb{R})$ define $F_\lambda(\vec{f}) := Se^{i\lambda(Z_t^H - c)}(\vec{f})$. Then we have

$$\begin{aligned}
F_\lambda(\vec{f}) &= \left\langle\left\langle \exp\left(i\lambda\left(\left\langle \cdot, \sum_{k=1}^n \alpha_k N_-^{H_k} \mathbf{1}_{[0,t]}\right\rangle - c\right)\right), : e^{\langle \cdot, \vec{f}\rangle} : \right\rangle\right\rangle \\
&= \exp\left(-ic\lambda - \frac{1}{2}\left|\vec{f}\right|_0^2\right) \int_{\mathcal{S}'_d(\mathbb{R})} \exp\left(\left\langle \vec{\omega}, i\lambda \sum_{k=1}^n \alpha_k N_-^{H_k} \mathbf{1}_{[0,t]} + \vec{f}\right\rangle\right) d\mu(\vec{\omega}) \\
&= \exp\left(-ic\lambda - \frac{1}{2}\left|\vec{f}\right|_0^2\right) \exp\left(\frac{1}{2}\left|\vec{f} + i\lambda \sum_{k=1}^n \alpha_k N_-^{H_k} \mathbf{1}_{[0,t]}\right|_0^2\right) \\
&= \exp\left(-ic\lambda - \frac{1}{2}|\lambda|^2 \sum_{k=1}^n \alpha_k^2 t^{2H_k}\right) \exp\left(i\lambda \int_{\mathbb{R}} \vec{f}(x) \sum_{k=1}^n \alpha_k N_-^{H_k} \mathbf{1}_{[0,t]}(x)\, dx\right).
\end{aligned}$$

In the last expression we have used the formula for the variance of GMFBM. The mapping $\lambda \mapsto F_\lambda(\vec{f})$ is measurable for all $\vec{f} \in \mathcal{S}_d(\mathbb{R})$ and $\lambda \in \mathbb{R}^d$. For the boundedness, let $z \in \mathbb{C}$:

$$\begin{aligned}
&\left|F_\lambda(z\vec{f})\right| \\
&= \left|\exp\left(-ic\lambda - \frac{1}{2}|\lambda|^2 \sum_{k=1}^n \alpha_k^2 t^{2H_k}\right)\right| \left|\exp\left(i\lambda \int_{\mathbb{R}} z\vec{f}(x) \sum_{k=1}^n \alpha_k N_-^{H_k} \mathbf{1}_{[0,t]}(x)\, dx\right)\right| \\
&\le \exp\left(-\frac{1}{2}|\lambda|^2 \sum_{k=1}^n \alpha_k^2 t^{2H_k}\right) \exp\left(|z||\lambda| \sum_{j=1}^d \sum_{k=1}^n |\alpha_k| C_{H_k} t \left(\sup_{x\in\mathbb{R}} |f_j(x)| + \sup_{x\in\mathbb{R}} |f'_j(x)| + |f_j|_0\right)\right),
\end{aligned}$$

where the last inequality follows from an application of Lemma 2 . Now define a continuous norm on $\mathcal{S}_d(\mathbb{R})$ as follow

$$\left\|\vec{f}\right\|_* := \sum_{j=1}^d \left(\sup_{x\in\mathbb{R}} |f_j(x)| + \sup_{x\in\mathbb{R}} |f'_j(x)| + |f_j|_0\right). \tag{4}$$

Hence

$$\begin{aligned}
\left|F_\lambda(z\vec{f})\right| &\le \exp\left(-\frac{1}{2}|\lambda|^2 \sum_{k=1}^n \alpha_k^2 t^{2H_k}\right) . \exp\left(|z||\lambda| \left\|\vec{f}\right\|_* \sum_{k=1}^n K_{H_k}\right) \\
&\le \exp\left(-\frac{1}{4}|\lambda|^2 \sum_{k=1}^n \alpha_k^2 t^{2H_k}\right) . \exp\left(\frac{\left(\sum_{k=1}^n K_{H_k}\right)^2}{\sum_{k=1}^n \alpha_k^2 t^{2H_k}} |z|^2 \left\|\vec{f}\right\|_*^2\right),
\end{aligned}$$

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where $K_{H_k} := |\alpha_k| C_{H_k} t$. The first factor is an integrable function of λ, and the second factor is constant with respect to λ. Therefore, according to the Corollary 4 $\delta_d\left(Z_t^H - c\right) \in (\mathcal{S})^*$. To obtain an explicit expression for the S-transform of $\delta_d\left(Z_t^H - c\right)$ we calculate as follow.

$$
\begin{aligned}
&S\left(\delta_d\left(Z_t^H - c\right)\right)(\vec{f}) \\
&= S\left(\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\lambda(Z_t^H - c)}\, d\lambda\right)(\vec{f}) \\
&= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} S e^{i\lambda(Z_t^H - c)}(\vec{f})\, d\lambda \\
&= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}|\lambda|^2 \sum_{k=1}^{n} \alpha_k^2 t^{2H_k} + i\lambda \int_{\mathbb{R}} \vec{f}(x) \sum_{k=1}^{n} \alpha_k N_-^{H_k} \mathbf{1}_{[0,t]}(x)\, dx\right) \\
&= \left(\frac{1}{2\pi}\right)^d \left(\frac{\pi}{\frac{1}{2}\sum_{k=1}^{n} \alpha_k^2 t^{2H_k}}\right)^{d/2} \exp\left(\frac{\left|i \int_R \vec{f}(x) \sum_{k=1}^{n} \alpha_k N_-^{H_k} \mathbf{1}_{[0,t]}(x)\, dx\right|_0^2}{2\sum_{k=1}^{n} \alpha_k^2 t^{2H_k}}\right) \\
&= \frac{1}{\left(2\pi \sum_{k=1}^{n} \alpha_k^2 t^{2H_k}\right)^{d/2}} \cdot \\
&\quad \exp\left(-\frac{1}{2\sum_{k=1}^{n} \alpha_k^2 t^{2H_k}} \sum_{j=1}^{d} \left(\int_{\mathbb{R}} \sum_{k=1}^{n} \alpha_k f_j(x) N_-^{H_k} \mathbf{1}_{[0,t]}(x)\, dx\right)^2\right). \blacksquare
\end{aligned}
$$

From the above result we can calculate easily the generalized expectation of the Donsker delta function of GMFBM, i.e.

$$
\mathbb{E}\left(\delta_d\left(Z_t^H - c\right)\right) = S\left(\delta_d\left(Z_t^H - c\right)\right)(0) = \frac{1}{\left(2\pi \sum_{k=1}^{n} \alpha_k^2 t^{2H_k}\right)^{d/2}}.
$$

4 Concluding Remarks

A review of the construction and analysis of the GMFBM within the white noise theory has been presented. We prove that Donsker's delta function of GMFBM is an object in the space of Hida distributions. As future work we are interested in the possible applications of the results obtained in this paper, such as local times and self-intersection local times.

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